ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

Journal of Computational and Applied Mathematics 208 (2007) 425–433

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

On some new nonlinear discrete inequalities and their applications☆

Fan Wei Meng*, Dehong Ji

Department of Mathematics, Qufu Normal University, Qufu 273165, PR China

Received 16 May 2005; received in revised form 3 April 2006

Abstract

In this paper, some new discrete inequalities in two independent variables which provide explicit bounds on unknown functions are established. The inequalities given here can be used as tools in the qualitative theory of certain finite difference equations.

MSC: primary 26D15; 26D20

Keywords: Discrete inequalities; Two independent variables; Difference equation

1. Introduction

The finite difference inequalities involving functions of one and more than one independent variables which provide explicit bounds on unknown functions play a fundamental role in the development of the theory of finite difference equations. During the past few years, many such new inequalities have been discovered, which are motivated by certain applications. For example, see [2–9] and the references therein. In the qualitative analysis of some classes of finite difference equations, the bounds provided by the earlier inequalities are inadequate and it is necessary to seek some new inequalities in order to achieve a diversity of desired goals. In this paper, we establish some new discrete inequalities involving functions of two independent variables. Our results generalize some results in [6].

2. Main results

In what follows, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0, \infty)$, $\mathbb{N}_0 = 0, 1, 2, \ldots$ are the given subsets of $\mathbb{R}$. We use the usual conventions that empty sums and products are taken to be 0 and 1, respectively. Throughout this paper, all the functions which appear in the inequalities are assumed to be real valued and all the sums involved exist on the respective domains of their definitions.

The following lemmas are useful to prove our main results.

☆ This research was supported in part by NSF of Shandong Grant Y2005A06.

* Corresponding author.

E-mail addresses: fwmeng@qfnu.edu.cn (F.W. Meng), jdh200298@eyou.com (D. Ji).

0377-0427/$ - see front matter
doi:10.1016/j.cam.2006.10.024

Lemma 1 (*Pachpatte [6]*). *Let $u(n), a(n)$ and $b(n)$ be nonnegative functions defined for $n \in \mathbb{N}_0$ with $a(n)$ not equivalent to zero.*

(1) *Assume that $a(n)$ is nondecreasing for $n \in \mathbb{N}_0$. If*

$$u(n) \leqslant a(n) + \sum_{s=0}^{n-1} b(s)u(s)$$

for $n \in \mathbb{N}_0$, then

$$u(n) \leqslant a(n) \prod_{s=0}^{n-1} [1 + b(s)]$$

for $n \in \mathbb{N}_0$.

(2) *Assume that $a(n)$ is nonincreasing for $n \in \mathbb{N}_0$. If*

$$u(n) \leqslant a(n) + \sum_{s=n+1}^{\infty} b(s)u(s)$$

for $n \in \mathbb{N}_0$, then

$$u(n) \leqslant a(n) \prod_{s=n+1}^{\infty} [1 + b(s)]$$

for $n \in \mathbb{N}_0$.

Lemma 2. *Assume that $p \geqslant q > 0$, $a \geqslant 0$, then*

$$a^{\frac{q}{p}} \leqslant \frac{q}{p} k^{\frac{q-p}{p}} a + \frac{p-q}{p} k^{\frac{q}{p}}$$

for any $k > 0$.

Proof. Let $b = \frac{p}{q}$, then $b \geqslant 1$, by [1, Lemma 2], we have

$$a^{\frac{q}{p}} \leqslant \frac{q}{p} k^{\frac{q-p}{p}} a + \frac{p-q}{p} k^{\frac{q}{p}},$$

for any $k > 0$. □

Theorem 3. *Let $u(m,n), a(m,n), b(m,n), c(m,n), d(m,n), e(m,n)$ be nonnegative functions defined for $m, n \in \mathbb{N}_0$ with $a(m,n)$ not equivalent to zero, and $p \geqslant q > 0$, $p \geqslant r > 0$, p, q, r are constants. If*

$$[u(m,n)]^p \leqslant a(m,n) + b(m,n) \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} [c(s,t)(u(s,t))^q + d(s,t)(u(s,t))^r + e(s,t)] \tag{1}$$

for $m, n \in \mathbb{N}_0$, then

$$u(m,n) \leqslant \Bigg[a(m,n) + b(m,n) f(m,n) \times \prod_{s=0}^{m-1} \left(1 + \sum_{t=n+1}^{\infty} \left(\frac{q}{p} k^{\frac{q-p}{p}} c(s,t) b(s,t) + \frac{r}{p} k^{\frac{r-p}{p}} d(s,t) b(s,t) \right) \right) \Bigg]^{\frac{1}{p}}, \tag{2}$$

for any $k>0$, $m,n\in\mathbb{N}_0$, *where*

$$f(m,n)=\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\left[c(s,t)\left(\frac{p-q}{p}k^{\frac{q}{p}}+a(s,t)\frac{q}{p}k^{\frac{q-p}{p}}\right)\right.$$
$$\left.+\,d(s,t)\left(\frac{p-r}{p}k^{\frac{r}{p}}+a(s,t)\frac{r}{p}k^{\frac{r-p}{p}}\right)+e(s,t)\right],\tag{3}$$

for $m,n\in\mathbb{N}_0$, $k>0$.

Proof. Define a function $z(m,n)$ by

$$z(m,n)=\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}[c(s,t)(u(s,t))^q+d(s,t)(u(s,t))^r+e(s,t)].\tag{4}$$

Then (1) can be restated as

$$[u(m,n)]^p\leqslant a(m,n)+b(m,n)z(m,n).\tag{5}$$

By (5), we have

$$u(m,n)\leqslant(a(m,n)+b(m,n)z(m,n))^{\frac{1}{p}}.\tag{6}$$

Thus, from (4), (6) we obtain

$$z(m,n)\leqslant\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}[c(s,t)(a(s,t)+b(s,t)z(s,t))^{\frac{q}{p}}+d(s,t)(a(s,t)+b(s,t)z(s,t))^{\frac{r}{p}}+e(s,t)].\tag{7}$$

By Lemma 2, we have

$$\begin{aligned}z(m,n)&\leqslant\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\left[c(s,t)\left(\frac{q}{p}k^{\frac{q-p}{p}}(a(s,t)+b(s,t)z(s,t))+\frac{p-q}{p}k^{\frac{q}{p}}\right)\right.\\&\quad\left.+d(s,t)\left(\frac{r}{p}k^{\frac{r-p}{p}}(a(s,t)+b(s,t)z(s,t))+\frac{p-r}{p}k^{\frac{r}{p}}\right)+e(s,t)\right]\\&=f(m,n)+\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\left(\frac{q}{p}k^{\frac{q-p}{p}}c(s,t)b(s,t)+\frac{r}{p}k^{\frac{r-p}{p}}d(s,t)b(s,t)\right)z(s,t),\end{aligned}\tag{8}$$

where $f(m,n)$ is defined by (3). It is easy to see that $f(m,n)$ is nonnegative, continuous, nondecreasing in m and nonincreasing in n for $m,n\in\mathbb{N}_0$.

Firstly, we assume that $f(m,n)>0$ for $m,n\in\mathbb{N}_0$. From (8) we easily observe that

$$\frac{z(m,n)}{f(m,n)}\leqslant 1+\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\left(\frac{q}{p}k^{\frac{q-p}{p}}c(s,t)b(s,t)+\frac{r}{p}k^{\frac{r-p}{p}}d(s,t)b(s,t)\right)\frac{z(s,t)}{f(s,t)}.\tag{9}$$

Set

$$v(m,n)=1+\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\left(\frac{q}{p}k^{\frac{q-p}{p}}c(s,t)b(s,t)+\frac{r}{p}k^{\frac{r-p}{p}}d(s,t)b(s,t)\right)\frac{z(s,t)}{f(s,t)},\tag{10}$$

then

$$\frac{z(m,n)}{f(m,n)}\leqslant v(m,n).\tag{11}$$

From (10), we get

$$\begin{aligned}
&[v(m+1,n)-v(m,n)]-[v(m+1,n+1)-v(m,n+1)] \\
&\quad=\left(\frac{q}{p}k^{\frac{q-p}{p}}c(m,n+1)b(m,n+1)+\frac{r}{p}k^{\frac{r-p}{p}}d(m,n+1)b(m,n+1)\right)\frac{z(m,n+1)}{f(m,n+1)} \\
&\quad\leqslant\left(\frac{q}{p}k^{\frac{q-p}{p}}c(m,n+1)b(m,n+1)+\frac{r}{p}k^{\frac{r-p}{p}}d(m,n+1)b(m,n+1)\right)v(m,n+1).
\end{aligned}\tag{12}$$

From (11) and using the fact that $v(m,n)>0$, $v(m,n+1)\leqslant v(m,n)$ for $m,n\in\mathbb{N}_0$, we obtain

$$\begin{aligned}
&\frac{v(m+1,n)-v(m,n)}{v(m,n)}-\frac{v(m+1,n+1)-v(m,n+1)}{v(m,n+1)} \\
&\quad\leqslant\frac{q}{p}k^{\frac{q-p}{p}}c(m,n+1)b(m,n+1)+\frac{r}{p}k^{\frac{r-p}{p}}d(m,n+1)b(m,n+1).
\end{aligned}\tag{13}$$

Keeping m fixed in (12), setting $n=t$ and summing over $t=n,n+1,\ldots,\ r-1$, where $r\geqslant n+1$ is an arbitrary number in $\mathbb{N}_0$, then we obtain

$$\begin{aligned}
&\frac{v(m+1,n)-v(m,n)}{v(m,n)}-\frac{v(m+1,r)-v(m,r)}{v(m,r)} \\
&\quad\leqslant\sum_{t=n+1}^{r}\left(\frac{q}{p}k^{\frac{q-p}{p}}c(m,t)b(m,t)+\frac{r}{p}k^{\frac{r-p}{p}}d(m,t)b(m,t)\right).
\end{aligned}\tag{14}$$

Noticing that

$$\lim_{r\to\infty}v(m,r)=\lim_{r\to\infty}v(m+1,r)=1,$$

and letting $r\to\infty$ in (14), we get

$$\frac{v(m+1,n)-v(m,n)}{v(m,n)}\leqslant\sum_{t=n+1}^{\infty}\left(\frac{q}{p}k^{\frac{q-p}{p}}c(m,t)b(m,t)+\frac{r}{p}k^{\frac{r-p}{p}}d(m,t)b(m,t)\right),\tag{15}$$

i.e.,

$$v(m+1,n)\leqslant\left[1+\sum_{t=n+1}^{\infty}\left(\frac{q}{p}k^{\frac{q-p}{p}}c(m,t)b(m,t)+\frac{r}{p}k^{\frac{r-p}{p}}d(m,t)b(m,t)\right)\right]v(m,n).\tag{16}$$

Now by keeping n fixed in (16) setting $m=s$ and substituting $s=0,1,2,\ldots,m-1$ successively and then using the fact that $v(0,n)=1$, we have:

$$v(m,n)\leqslant\prod_{s=0}^{m-1}\left[1+\sum_{t=n+1}^{\infty}\left(\frac{q}{p}k^{\frac{q-p}{p}}c(s,t)b(s,t)+\frac{r}{p}k^{\frac{r-p}{p}}d(s,t)b(s,t)\right)\right].\tag{17}$$

From (11) and (17), we obtain

$$z(m,n)\leqslant f(m,n)\prod_{s=0}^{m-1}\left[1+\sum_{t=n+1}^{\infty}\left(\frac{q}{p}k^{\frac{q-p}{p}}c(s,t)b(s,t)+\frac{r}{p}k^{\frac{r-p}{p}}d(s,t)b(s,t)\right)\right].\tag{18}$$

The desired inequality (2) follows from (6) and (18).

If $f(m,n)$ is nonnegative, we carry out the above procedure with $f(m,n)+\varepsilon$ instead of $f(m,n)$ where $\varepsilon>0$ is an arbitrary small constant and subsequently pass to the limit as $\varepsilon\to 0$ to obtain (2). This completes the proof. $\square$

Theorem 4. *Let $u(m,n)$, $a(m,n)$, $b(m,n)$, $c(m,n)$, $d(m,n)$, $e(m,n)$ be nonnegative functions defined for $m,n \in \mathbb{N}_0$ with $a(m,n)$ not equivalent to zero, and $p \geqslant q > 0$, $p \geqslant r > 0$, p, q, r are constants. If*

$$[u(m,n)]^p \leqslant a(m,n) + b(m,n) \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} [c(s,t)(u(s,t))^q + d(s,t)(u(s,t))^r + e(s,t)], \tag{19}$$

for $m,n \in \mathbb{N}_0$, then

$$u(m,n) \leqslant \left[a(m,n) + b(m,n) \bar{f}(m,n) \times \prod_{s=m+1}^{\infty} \left(1 + \sum_{t=n+1}^{\infty} \left(\frac{q}{p} k^{\frac{q-p}{p}} c(s,t) b(s,t) + \frac{r}{p} k^{\frac{r-p}{p}} d(s,t) b(s,t) \right) \right) \right]^{\frac{1}{p}}, \tag{20}$$

for any $k > 0$, $m,n \in \mathbb{N}_0$, where

$$\bar{f}(m,n) = \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \left[c(s,t) \left(\frac{p-q}{p} k^{\frac{q}{p}} + a(s,t) \frac{q}{p} k^{\frac{q-p}{p}} \right) + d(s,t) \left(\frac{p-r}{p} k^{\frac{r}{p}} + a(s,t) \frac{r}{p} k^{\frac{r-p}{p}} \right) + e(s,t) \right] \tag{21}$$

for $m,n \in \mathbb{N}_0$.

The proof of Theorem 4 can be completed by following the proof of Theorem 3 with suitable changes, we omit it here.

Remark 5. If we take $q = 1$, $d(m,n) = 0$ or $r = 1$, $c(m,n) = 0$, then the inequalities established in Theorems 3 and 4 reduce to the inequalities established in [1, Theorems 1 and 2].

Remark 6. If we take $q = 1$, $d(m,n) = 0$ or $r = 1$, $c(m,n) = 0$, and $p = 1$, $e(m,n) = 0$, then the inequalities established in Theorems 3 and 4 reduce to the inequalities established in [6, Theorem 2.6 ($puuu1$) and ($puuu2$)].

Theorem 7. *Let $u(m,n)$, $a(m,n)$, $b(m,n)$, $c(m,n)$, $d(m,n)$, $e(m,n)$ be nonnegative functions defined for $m,n \in \mathbb{N}_0$ with $a(m,n)$ not equivalent to zero. Assume that $a(m,n)$ be nondecreasing in $m \in \mathbb{N}_0$, and $p \geqslant q > 0$, $p \geqslant r > 0$, p, q, r are constants. If*

$$[u(m,n)]^p \leqslant a(m,n) + \sum_{s=0}^{m-1} b(s,n)[u(s,n)]^p + \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} [c(s,t)(u(s,t))^q + d(s,t)(u(s,t))^r + e(s,t)], \tag{22}$$

for $m,n \in \mathbb{N}_0$, then

$$u(m,n) \leqslant (r(m,n))^{\frac{1}{p}} \left[a(m,n) + F(m,n) \times \prod_{s=0}^{m-1} \left(1 + \sum_{t=n+1}^{\infty} \frac{q}{p} k^{\frac{q-p}{p}} c(s,t)(r(s,t))^{\frac{q}{p}} + \frac{r}{p} k^{\frac{r-p}{p}} d(s,t)(r(s,t))^{\frac{r}{p}} \right) \right]^{\frac{1}{p}}, \tag{23}$$

for any $k>0$, $m,n\in\mathbb{N}_0$, where

$$F(m,n)=\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\left[c(s,t)(r(s,t))^{\frac{q}{p}}\left(\frac{p-q}{p}k^{\frac{q}{p}}+a(s,t)\frac{q}{p}k^{\frac{q-p}{p}}\right)\right.$$
$$\left.+d(s,t)(r(s,t))^{\frac{r}{p}}\left(\frac{p-r}{p}k^{\frac{r}{p}}+a(s,t)\frac{r}{p}k^{\frac{r-p}{p}}\right)+e(s,t)\right],\tag{24}$$

$$r(m,n)=\prod_{s=0}^{m-1}[1+b(s,n)]\tag{25}$$

for $m,n\in\mathbb{N}_0$.

Proof. Define a function $z(m,n)$ by

$$z(m,n)=a(m,n)+\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}[c(s,t)(u(s,t))^q+d(s,t)(u(s,t))^r+e(s,t)].\tag{26}$$

Then (22) can be restated as

$$[u(m,n)]^p\leqslant z(m,n)+\sum_{s=0}^{m-1}b(s,n)[u(s,n)]^p.\tag{27}$$

Clearly, $z(m,n)$ is a nonnegative and nondecreasing function in m, $m\in\mathbb{N}_0$. Treating n, $n\in\mathbb{N}_0$ fixed in (27), and using Lemma 1(1) to (27), we have

$$[u(m,n)]^p\leqslant z(m,n)r(m,n),\tag{28}$$

where $r(m,n)$ is defined by (24). From (28) and (26) we obtain:

$$[u(m,n)]^p\leqslant r(m,n)(a(m,n)+v(m,n)),\tag{29}$$

where

$$v(m,n)=\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}[c(s,t)(u(s,t))^q+d(s,t)(u(s,t))^r+e(s,t)].\tag{30}$$

From (29), we have

$$u(m,n)\leqslant(r(m,n))^{\frac{1}{p}}(a(m,n)+v(m,n))^{\frac{1}{p}},\tag{31}$$

for $m,n\in\mathbb{N}_0$. From (30), (31) and Lemma 2, we get

$$v(m,n)\leqslant\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}[c(s,t)(r(s,t))^{\frac{q}{p}}(a(s,t)+v(s,t))^{\frac{q}{p}}+d(s,t)(r(s,t))^{\frac{r}{p}}(a(s,t)+v(s,t))^{\frac{r}{p}}+e(s,t)]$$
$$\leqslant\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\left[c(s,t)(r(s,t))^{\frac{q}{p}}\left(\frac{p-q}{p}k^{\frac{q}{p}}+\frac{q}{p}k^{\frac{q-p}{p}}(a(s,t)+v(s,t))\right)\right.$$
$$\left.+d(s,t)(r(s,t))^{\frac{r}{p}}\left(\frac{p-r}{p}k^{\frac{r}{p}}+\frac{r}{p}k^{\frac{r-p}{p}}(a(s,t)+v(s,t))\right)+e(s,t)\right]$$
$$=F(m,n)+\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\left[c(s,t)(r(s,t))^{\frac{q}{p}}\frac{q}{p}k^{\frac{q-p}{p}}v(s,t)+d(s,t)(r(s,t))^{\frac{r}{p}}\frac{r}{p}k^{\frac{r-p}{p}}v(s,t)\right],\tag{32}$$

for $m, n \in \mathbb{N}_0$, $k > 0$, where $F(m,n)$ is defined by (25). The rest of the proof of (23) can be completed by following the proof of Theorem 3, we omit the details. □

Theorem 8. *Let $u(m,n)$, $a(m,n)$, $b(m,n)$, $c(m,n)$, $d(m,n)$, $e(m,n)$ be nonnegative functions defined for $m, n \in \mathbb{N}_0$ with $a(m,n)$ not equivalent to zero. Assume that $a(x,y)$ be nonincreasing in $m \in \mathbb{N}_0$, and $p \geqslant q > 0$, $p \geqslant r > 0$, p, q, r are constants. If*

$$[u(m,n)]^p \leqslant a(m,n) + \sum_{s=m+1}^{\infty} b(s,n)[u(s,n)]^p + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} [c(s,t)(u(s,t))^q + d(s,t)(u(s,t))^r + e(s,t)], \tag{33}$$

for $m, n \in \mathbb{N}_0$, then

$$u(m,n) \leqslant (\bar{r}(m,n))^{\frac{1}{p}} \left[a(m,n) + \bar{F}(m,n) \times \prod_{s=m+1}^{\infty} \left(1 + \sum_{t=n+1}^{\infty} \frac{q}{p} k^{\frac{q-p}{p}} c(s,t)(\bar{r}(s,t))^{\frac{q}{p}} + \frac{r}{p} k^{\frac{r-p}{p}} d(s,t)(\bar{r}(s,t))^{\frac{r}{p}} \right) \right]^{\frac{1}{p}}, \tag{34}$$

for any $k > 0$, $m, n \in \mathbb{N}_0$, where

$$\bar{F}(m,n) = \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} \left[c(s,t)(\bar{r}(s,t))^{\frac{q}{p}} \left(\frac{p-q}{p} k^{\frac{q}{p}} + a(s,t)\frac{q}{p} k^{\frac{q-p}{p}} \right) + d(s,t)(\bar{r}(s,t))^{\frac{r}{p}} \left(\frac{p-r}{p} k^{\frac{r}{p}} + a(s,t)\frac{r}{p} k^{\frac{r-p}{p}} \right) + e(s,t) \right], \tag{35}$$

$$\bar{r}(m,n) = \prod_{s=m+1}^{\infty} [1 + b(s,n)], \tag{36}$$

for $m, n \in \mathbb{R}_+$.

The proof of Theorem 8 can be completed by following the proof of Theorem 7 with suitable changes, we omit it here.

Remark 9. If we take $q = 1$, $d(m,n) = 0$ or $r = 1$, $c(m,n) = 0$, then the inequalities established in Theorems 7 and 8 reduce to the inequalities established in [1, Theorems 1 and 4].

Remark 10. If we take $q = 1$, $d(m,n) = 0$ or $r = 1$, $c(m,n) = 0$, and $p = 1$, $e(m,n) = 0$, then the inequalities established in Theorems 7 and 8 reduce to the inequalities established in [6, Theorem 2.7].

3. Some applications

Example 11. Consider the finite difference equation:

$$[u(m,n)]^p = a(m,n) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} h(s,t,u(s,t)), \tag{37}$$

where $h: \mathbb{N}_0^2 \times \mathbb{R} \to \mathbb{R}$, $a: \mathbb{N}_0^2 \to \mathbb{R}$.

Suppose that

$$|h(m,n,u)| \leqslant c(m,n)|u|^q, \tag{38}$$

where $c(m,n)$ are nonnegative continuous functions for $m,n \in \mathbb{N}_0$, $p \geqslant q > 0$, p, q, are constants. If $u(m,n)$ is any solution of (37), then

$$|u(m,n)| \leqslant \left[|a(m,n)| + \bar{f}(m,n) \prod_{s=m+1}^{\infty} \left(1 + \sum_{t=n+1}^{\infty} \frac{q}{p} k^{\frac{q-p}{p}} c(s,t) \right) \right]^{\frac{1}{p}}, \tag{39}$$

for $m,n \in \mathbb{N}_0$, $k > 0$, where

$$\bar{f}(m,n) = \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \left[c(s,t) \left(\frac{p-q}{p} k^{\frac{q}{p}} + \frac{q}{p} k^{\frac{q-p}{p}} |a(s,t)| \right) \right], \tag{40}$$

for $m,n \in \mathbb{N}_0$, $k > 0$.

In fact, if $u(m,n)$ is any solution of (37), then it satisfies the equivalent equation:

$$|u(m,n)|^p \leqslant |a(m,n)| + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} c(s,t)|u(s,t)|^q. \tag{41}$$

Now a suitable application of the inequality in Theorem 4 to (41) yields (39).

Example 12. Consider the finite difference equation

$$u(m,n) - u(m,n-1) - u(m-1,n) + u(m-1,n-1) = h(m,n,u(m,n)) + r(m,n), \tag{42}$$

$$u(m,\infty) = \sigma(m), \quad u(\infty,n) = \tau(n), \quad u(\infty,\infty) = d, \tag{43}$$

where $h: \mathbb{N}_0^2 \times \mathbb{R} \to \mathbb{R}$, $r: \mathbb{N}_0^2 \to \mathbb{R}$, σ, $\tau: \mathbb{N}_0 \to \mathbb{R}$, d is a real constant.

Suppose that

$$|h(m,n,u) - h(m,n,v)| \leqslant c(m,n)|u-v|^q, \tag{44}$$

where $c(m,n)$ is defined as in Theorem 2, $q \leqslant 1$, q is a constant.

If $u(m,n)$, $v(m,n)$ are two solutions of (42)–(43), then

$$|u(m,n) - v(m,n)| \leqslant \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} (c(s,t)(1-q)k^q) \prod_{s=m+1}^{\infty} \left(1 + \sum_{t=n+1}^{\infty} q k^{q-1} c(s,t) \right), \tag{45}$$

for $m,n \in \mathbb{N}_0$, $k > 0$.

In fact, if $u(m,n)$ is a solution of (42)–(43), then it can be written as

$$u(m,n) = \sigma(m) + \tau(n) - d + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} [h(s,t,u(s,t)) + r(s,t)], \tag{46}$$

let $u(m,n)$, $v(m,n)$ be two solutions of (42)–(43), we have:

$$|u(m,n) - v(m,n)| \leqslant \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} c(s,t)|u(s,t) - v(s,t)|^q, \tag{47}$$

for $m,n \in \mathbb{N}_0$.

Now a suitable application of the inequality in Theorem 4 to (47) yields (45).

Acknowledgment

The authors thank the referees for their help to improve this paper.

References

[1] F.W. Meng, W.N. Li, On some new nonlinear discrete inequalities and their applications, J. Comput. Appl. Math. 158 (2003) 407–417.
[2] B.G. Pachpatte, On certain new finite difference inequalities, Indian, J. Pure Appl. Math. 24 (1993) 373–384.
[3] B.G. Pachpatte, Some new finite difference inequalities, Comput. Math. Appl. 28 (1994) 227–241.
[4] B.G. Pachpatte, On some new discrete inequalities useful in the theory of partial finite difference equations, Ann. Differential Equations 12 (1996) 1–12.
[5] B.G. Pachpatte, Inequalities applicable in the theory of finite difference equations, J. Math. Anal. Appl. 222 (1998) 438–459.
[6] B.G. Pachpatte, On some fundamental integral inequalities and their discrete analogues, J. Ineq. Pure Appl. Math. 2 (2001) Article 15.
[7] B.G. Pachpatte, Inequalities for Finite Difference Equations, Marcel Dekker, New York, 2002.
[8] E.H. Yang, Generalizations of Pachpatte's integral and discrete inequalities, Ann. Differential Equations 13 (1997) 180–188.
[9] E.H. Yang, A new integral inequality with power nonlinear and its discrete analogue, Acta Math. Appl. Sinica 17 (2001) 233–239.